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Transition from chaotic to regular behaviour in superposed matrix ensemble

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Abstract : An expression has been derived for the probability density function of the nearest level spacing using superposed matrix ensemble In one limit, it gives Poisson distribution and in the other, Wigner distribution The explicit form of the spacing distribution is Brody distribution which was given earlier by Brody empirically

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Matrix ensembles have recently been used in the study of Quantum chaos Several new matrix ensembles [1] in which the joint probability density function of the Hamiltonian matrix elements is non-invariant under rotations, have been introduced for this purpose. The main interest is to see how the spacing distribution changes when the distribution of the Hamiltonian matrix elements becomes non-invariant It becomes increasingly more difficult to derive analytic results as the dimension of the matrix starts increasing. An interesting problem is to see what kind of probability density function one finds for the nearest level spacing In the present work, we show that a superposed matrix ensemble can be constructed to study the transition from chaotic to regular behaviour. We shall first describe the two-dimensional ensemble and then show how to generalize to an arbitrary $N \times N$ dimension

Let us consider a 2 × 2 real-symmetric Hamiltonian matrix whose elements are denoted by $H_{\mu\nu}$. The joint probability density function $P(\{H_{\mu\nu}\})$ in superposed matrix ensemble is given by

$$P(\{H_{\mu\nu}\}) = \int d\alpha g(\alpha)\exp(-\alpha \operatorname{Tr} H^2), \tag{1}$$

where Tr denotes the trace of the matrix $g(\alpha)$ is the weight function, the form of which will be given later. Transforming to eigenvalue E_μ in the usual way [2] and writing the spacing $S = |E_1 - E_2|$, it is a simple matter to show that the probability density function of S is given by

$$p(S) = \sqrt{2\pi}S\int d\alpha \frac{g(\alpha)}{\sqrt{\alpha}} \exp\left(-\frac{\alpha}{2}S^2\right) \tag{2}$$

Writing $\alpha = x^2$, we can rewrite (2) as

$$p(S) = 2\sqrt{2\pi}S\int dx g(x^2)\exp\left(-\frac{1}{2}x^2S^2\right) \tag{3}$$

We have now to choose $g(x^2)$, so that in one limit, we get Poisson distribution and in the other limit, Wigner distribution From the integral [3]

$$\int_0^\infty dt \exp\left(-at^2 - \frac{b}{t^2}\right) = \frac{1}{2}\sqrt{\frac{\pi}{a}} \exp -2\sqrt{ab}. \tag{4}$$

we see that if $g(x^2)$ is taken to be

$$g(x^2) = \exp\left(-\frac{1}{2x^2}\right), \tag{5}$$

then integration of expression (3) between 0 and ∞ gives

$$p(S) = 2\pi\exp(-S), \tag{6}$$

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which is Poisson distribution. $p(S)$ as given by expression (6) is not normalized to unity. The normalized $p(S)$ is given by

$$p(S)=\exp(-S). \tag{7}$$

Wigner distribution is easy to get from expression (3) if we take

$$g(x^2)=\frac{1}{\sqrt{2\pi}}\delta(x^2-1). \tag{8}$$

This gives the normalized Wigner distribution

$$p(S)=S\exp\quad S^2 \tag{9}$$

The forms of $g(x^2)$ given by expressions (6) and (8) which give $p(S)$ in the two limits suggest that $g(x^2)$ should in general be of the form

$$g(x^2)=x^{-2\lambda}\exp(-ax^{-2\mu}), \tag{10}$$

where λ, a, μ are parameters. If we use this form of $g(x^2)$ in expression (3), we cannot write the exact integral of x between 0 and ∞. We approximately evaluate the integral using method of Steepest descent. Choosing λ, a from the two limits we find that the probability density function $p(S)$ from Poisson to Wigner limit is given by

$$p(S)=NS^{\frac{\mu-1}{\mu+1}}\exp\ -2\left(\frac{S}{2}\right)^{\frac{2\mu}{\mu+1}} \tag{11}$$

where N is the normalization constant. Introducing a new parameter $\beta(0\leq\beta\leq 1)$, $\beta=\frac{1}{\mu}$ and putting the value of N, we finally get

$$p(S)=\frac{2^{\frac{2\beta}{1+\beta}}}{1+\beta}S^{\frac{1-\beta}{1+\beta}}\exp\quad 2\left(\frac{S}{2}\right)^{\frac{2}{1+\beta}} \tag{12}$$

We would now like to make a couple of remarks about the method of steepest descent which has been used to derive expression (11). First, the accuracy of the method can easily be checked by evaluating the known integral given by expression (4) using steepest descent. One could find the maxima of $at^2+\frac{b}{t^2}$ and expand the exponent around this maxima. The first two terms of the expansion give the same value of the integral as the one given by expression (11). This means that the corrections arising due to higher terms cancel each other. Thus, the method of steepest descent provides an accurate method to evaluate integrals of the form given by expressions (3) and (10).

The second remark is that an important result of matrix ensemble theory is that once the dominant part of probability density is found, one could take care of any deviations by constructing orthogonal polynomials using the dominant part as weight function.

Before the study of chaos, it was found by Brody [4] that one could get a better fit to $p(S)$ if one uses expression (12), which is called Brody distribution. The present formulation shows that Brody distribution is a natural consequence of superposed matrix ensemble.

We shall now discuss how to generalize the results for $N\times N$ Hamiltonian matrix. For this case, one again starts from expression (1), except that Tr is now a trace in N dimensions. Transformation [2] to eigenvalues E_μ gives after integrating over eigenvectors,

$$P(\{E_\mu\})=\int d\alpha g(\alpha)\exp\left(-\alpha\sum_{\mu=1}^{N}E_\mu^2\right.$$

$$\prod_{\mu<\nu}|E_\mu-E_\nu| \tag{13}$$

Now, it is known [2] that the exact spacing distribution which one gets from Wishart distribution $\left[\exp\left(-\alpha\Sigma E_\mu^2\right)\right]$ $\times\prod_{\mu<\nu}|E_\mu-E_\nu|$ is very close to the one given by Wigner distribution $S\exp\left(-\frac{1}{2}\alpha S^2\right)$ when dimension N is large. This shows that one can again write $p(S)$ in the form given by expression (2) which by choosing $g(x^2)$ as in expression (10) will give $p(S)$ as given by expression (12). Thus, $N\times N$ superposed matrix ensemble gives the probability density function of the spacing S given by expression (12).

We now conclude with the following remarks. The first is that in the superposed matrix ensemble, there are strong correlations between Hamiltonian matrix elements $H_{\mu\nu}$. Correlations between Hamiltonian matrix elements were introduced in a different way by Cheon [1]. By choosing the weight function $g(\alpha)$, we have been able to describe the transition from chaotic to regular behaviour The formulation also provides a basis for the derivation of Brody's distribution which was only an empirical distribution so far.

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